Communication

Explicit Semianalytical Expressions of Sensitivity Matrices for the Reconstruction of 1-D Planarly Layered TI Media Illuminated by 3-D Sources

Feng Han, Jianliang Zhuo, Shan Lu, Jianwen Wang, and Qing Huo Liu

***Abstract*—In this communication, the sensitivity matrices for the reconstruction of 1-D planarly layered transverse isotropic (TI) media illuminated by electromagnetic wave excited by 3-D electric or magnetic current sources are derived based on the transmission-line network analogy. The elements of the matrices are expressed analytically in the vertical *z*-direction but are in the form of Sommerfeld integration in the horizontal xy-plane. The derived expressions are useful for the inversion of dielectric parameters and boundary positions of 1-D layered TI media.**

***Index Terms*—Planarly layered media, sensitivity matrices, transversally isotropic.**

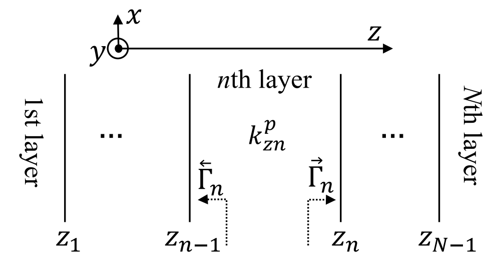

Fig. 1. Geometry structure of the 1-D planarly layered medium.

I. Introduction

Electromagnetic (EM) scattering and inverse scattering are ubiquitous in modern civilian and military activities. The major applications of EM inverse scattering include microwave imaging [1], radar remote sensing [2], geophysical exploration [3], and so on. The forward scattering is to map the model parameters to receiver responses, while the inverse scattering is to infer the model parameters from receiver responses. For a rigorous EM inversion problem, its intrinsic nonlinearity always requires iterations. As a result, the sensitivity matrix of the data vector with respect to the vector of model parameters, that is, the first-order derivative of the forward computation model to the model parameters, is of great importance. In literature, the sensitivity matrix is also called Fréchet derivative matrix or Jacobian matrix.

In many EM inversion applications such as ground-penetrating radar (GPR) [4], airborne transient EMs (TEMs) [5], semiairborne TEM [3], logging-while-drilling (LWD) [6], and the induction logging [7], the simplest model of the underground structure is the 1-D planarly layered medium. Meanwhile, the uniaxial anisotropy (also called transverse isotropy) is always taken into account. However, the sources are 3-D. Consequently, the evaluation of the sensitivity matrix for 1-D layered media illuminated by 3-D sources requires integration in the horizontal xy-plane. Related results have been presented in several previous works. For example, in [3], the Fréchet derivative was derived for a horizontal electric line source. However, it is only valid for isotropic media and not applicable to uniaxial anisotropy. In [7] and [8], partial semianalytical expressions of the Fréchet derivatives in the double-integral forms were given for a magnetic dipole source.

In this communication, we present the semianalytical expressions of the sensitivity matrices for both electric and magnetic sources with the arbitrary polarization based on the transmission-line network analogy [9]. The analytical expressions in the z-direction are explicitly given. In the horizontal xy-plane, the expressions are in the form of Sommerfeld integration which can be efficiently computed using the Gaussian quadratures [10]. This communication is organized as follows. In Section II, the problem is formulated. The sensitivity matrices for electric sources and magnetic sources are derived in Sections III and IV, respectively. The analytical solutions of the integration of dz are given in Section V. In Section VI, the sensitivity matrices for the layer boundaries are derived. In Section VII, the implementation of numerical integration is discussed. Finally, in Section VIII, the conclusion is drawn.

II. Problem Formulation

As shown in Fig. 1, the transverse isotropic (TI) medium is 1-D layered and the boundary position of each layer is denoted by z_n. The electric and magnetic sources can be placed inside an arbitrary layer. Maxwell's equations with the electric and magnetic sources $\mathbf{J}$ and $\mathbf{M}$ are expressed as

$$\nabla \times \mathbf{E} = -j\omega\overline{\overline{\mu}}_r\mu_0\mathbf{H} - \mathbf{M} \tag{1a}$$
$$\nabla \times \mathbf{H} = j\omega\overline{\overline{\epsilon}}_r\varepsilon_0\mathbf{E} + \mathbf{J} \tag{1b}$$

where $\overline{\overline{\epsilon}}_r$ is the complex relative permittivity and expressed as

$$\overline{\overline{\epsilon}}_r = \mathrm{diag}\{\epsilon_h, \epsilon_h, \epsilon_v\} = \mathrm{diag}\{\varepsilon_h, \varepsilon_h, \varepsilon_v\} + \frac{1}{j\omega\varepsilon_0}\mathrm{diag}\{\sigma_h, \sigma_h, \sigma_v\} \tag{2a}$$
$$\overline{\overline{\mu}}_r = \mathrm{diag}\{\mu_h, \mu_h, \mu_v\} \tag{2b}$$

where ϵ is a complex number and ε is a real number. The $\mathbf{E}$- and $\mathbf{H}$ fields at a receiver can be evaluated by

$$\mathbf{E} = \int_V \overline{\overline{\mathbf{G}}}_{\mathbf{EJ}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}' + \int_V \overline{\overline{\mathbf{G}}}_{\mathbf{EM}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{M}(\mathbf{r}')d\mathbf{r}' \tag{3a}$$
$$\mathbf{H} = \int_V \overline{\overline{\mathbf{G}}}_{\mathbf{HJ}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}' + \int_V \overline{\overline{\mathbf{G}}}_{\mathbf{HM}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{M}(\mathbf{r}')d\mathbf{r}' \tag{3b}$$

where $\overline{\overline{\mathbf{G}}}_{\mathbf{EJ}}$, $\overline{\overline{\mathbf{G}}}_{\mathbf{EM}}$, $\overline{\overline{\mathbf{G}}}_{\mathbf{HJ}}$, and $\overline{\overline{\mathbf{G}}}_{\mathbf{HM}}$ are the layered medium dyadic Green's functions (DGFs) and derived based on the transmission-line network analogy [9]. Now, assume the $\overline{\overline{\epsilon}}_r$ in the nth layer has a small perturbation $\boldsymbol{\delta}\overline{\overline{\epsilon}}_r = \mathrm{diag}\{\delta\epsilon_h, \delta\epsilon_h, \delta\epsilon_v\}$ and this causes the field

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perturbations $\boldsymbol{\delta}\mathbf{E}$ and $\boldsymbol{\delta}\mathbf{H}$ at the receiver. They can be evaluated by taking variations of two sides of (1)

$$\nabla \times \boldsymbol{\delta}\mathbf{E} = -j\omega\overline{\overline{\mu}}_r\mu_0\boldsymbol{\delta}\mathbf{H} \tag{4a}$$

$$\nabla \times \boldsymbol{\delta}\mathbf{H} = j\omega\overline{\overline{\epsilon}}_r\varepsilon_0\boldsymbol{\delta}\mathbf{E} + j\omega\boldsymbol{\delta}\overline{\overline{\epsilon}}_r\varepsilon_0\mathbf{E}. \tag{4b}$$

By comparing (1) and (4), we can easily find that the perturbations $\boldsymbol{\delta}\mathbf{E}$ and $\boldsymbol{\delta}\mathbf{H}$ at the receiver are excited by the equivalent electric source $\mathbf{J}' = j\omega\boldsymbol{\delta}\overline{\overline{\epsilon}}_r\varepsilon_0\mathbf{E}$ locating inside the nth layer. In EM inversion problems, the field perturbation caused by unit changes in model parameters, for example, the variation in dielectric parameter, is called the sensitivity matrix or Fréchet derivative. The field perturbations $\boldsymbol{\delta}\mathbf{E}$ and $\boldsymbol{\delta}\mathbf{H}$ can be compactly written as

$$\begin{aligned}\boldsymbol{\delta}\mathbf{E} &= \left(\mathbf{F}_{\mathbf{J}}^{\mathbf{E}} + \mathbf{F}_{\mathbf{M}}^{\mathbf{E}}\right) \cdot j\omega\varepsilon_0\boldsymbol{\delta}\overline{\overline{\epsilon}}_r \\ &= j\omega\varepsilon_0\left(\begin{bmatrix} F_{\mathbf{J},h}^{ex} & F_{\mathbf{J},v}^{ex} \\ F_{\mathbf{J},h}^{ey} & F_{\mathbf{J},v}^{ey} \\ F_{\mathbf{J},h}^{ez} & F_{\mathbf{J},v}^{ez}\end{bmatrix} + \begin{bmatrix} F_{\mathbf{M},h}^{ex} & F_{\mathbf{M},v}^{ex} \\ F_{\mathbf{M},h}^{ey} & F_{\mathbf{M},v}^{ey} \\ F_{\mathbf{M},h}^{ez} & F_{\mathbf{M},v}^{ez}\end{bmatrix}\right) \cdot \begin{bmatrix}\delta\epsilon_h \\ \delta\epsilon_v\end{bmatrix}\end{aligned} \tag{5a}$$

$$\begin{aligned}\boldsymbol{\delta}\mathbf{H} &= \left(\mathbf{F}_{\mathbf{J}}^{\mathbf{H}} + \mathbf{F}_{\mathbf{M}}^{\mathbf{H}}\right) \cdot j\omega\varepsilon_0\boldsymbol{\delta}\overline{\overline{\epsilon}}_r \\ &= j\omega\varepsilon_0\left(\begin{bmatrix} F_{\mathbf{J},h}^{hx} & F_{\mathbf{J},v}^{hx} \\ F_{\mathbf{J},h}^{hy} & F_{\mathbf{J},v}^{hy} \\ F_{\mathbf{J},h}^{hz} & F_{\mathbf{J},v}^{hz}\end{bmatrix} + \begin{bmatrix} F_{\mathbf{M},h}^{hx} & F_{\mathbf{M},v}^{hx} \\ F_{\mathbf{M},h}^{hy} & F_{\mathbf{M},v}^{hy} \\ F_{\mathbf{M},h}^{hz} & F_{\mathbf{M},v}^{hz}\end{bmatrix}\right) \cdot \begin{bmatrix}\delta\epsilon_h \\ \delta\epsilon_v\end{bmatrix}\end{aligned} \tag{5b}$$

where $\mathbf{F}_{\mathbf{J}}^{\mathbf{E}}$, $\mathbf{F}_{\mathbf{M}}^{\mathbf{E}}$, $\mathbf{F}_{\mathbf{J}}^{\mathbf{H}}$, and $\mathbf{F}_{\mathbf{M}}^{\mathbf{H}}$ are the sensitivity matrices for the perturbations of electric and magnetic fields at the receiver, respectively, when the excitation sources are electric and magnetic, respectively. One should note that each element of the matrix is the summation of three terms since $\mathbf{J}$ and $\mathbf{M}$ always have three components. For example, $F_{\mathbf{M},h}^{hy} = F_{Mx,h}^{hy} + F_{My,h}^{hy} + F_{Mz,h}^{hy}$ represents the sensitivity of H_y at the receiver with respect to the small perturbation of $\delta\epsilon_h$ in the nth layer when the medium is illuminated by the magnetic source $\mathbf{M} = \hat{x}M_x + \hat{y}M_y + \hat{z}M_z$. Therefore, there are totally 72 components to evaluate for the sensitivity matrices. In the following, we assume that the transmitter is an infinitesimal dipole source and locates in the m_sth layer, and its position is denoted as $\mathbf{r}_s = \hat{x}x_s + \hat{y}y_s + \hat{z}z_s$. The receiver locates in the m_rth layer and its position is denoted as $\mathbf{r}_r = \hat{x}x_r + \hat{y}y_r + \hat{z}z_r$. The nth layer has a perturbation of $\boldsymbol{\delta}\overline{\overline{\epsilon}}_r$ and an arbitrary field point inside it is denoted as $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$. In addition, there is a 2-D Fourier transform relationship between the spatial domain layered DGF and the spectral domain DGF which is written as [9]

$$\overline{\overline{\mathbf{G}}}_{\mathbf{EJ}}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi^2}\iint_{-\infty}^{+\infty}\tilde{\mathbf{G}}_{\mathbf{EJ}}(\mathbf{k}_\rho; z, z')e^{-j\mathbf{k}_\rho\cdot\boldsymbol{\rho}}dk_xdk_y \tag{6}$$

where $\tilde{\mathbf{G}}_{\mathbf{EJ}}$ is the spectral domain DGF, $\mathbf{r}$ and $\mathbf{r}'$ represent the field point and source point, respectively, $\boldsymbol{\rho} = \hat{x}(x - x') + \hat{y}(y - y')$ is the horizontal distance vector between the source point and the field point, and $\mathbf{k}_\rho = \hat{x}k_x + \hat{y}k_y$ is the horizontal wave vector. The other three DGFs in (3) have similar transforms.

III. Sensitivity Matrices Due to an Electric Dipole

In virtue of (3a) and (4), the electric field perturbation at $\mathbf{r}_r$ induced by the equivalent electric source $\mathbf{J}'$ in the nth layer is

$$\begin{aligned}\boldsymbol{\delta}\mathbf{E}(\mathbf{r_r}) &= \int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dxdy\overline{\overline{\mathbf{G}}}_{\mathbf{EJ}}(\mathbf{r_r},\mathbf{r})\cdot\mathbf{J}'(\mathbf{r}) \\ &= \int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dxdy\overline{\overline{\mathbf{G}}}_{\mathbf{EJ}}(\mathbf{r_r},\mathbf{r})\cdot j\omega\boldsymbol{\delta}\overline{\overline{\epsilon}}_r\varepsilon_0\mathbf{E}(\mathbf{r}).\end{aligned} \tag{7}$$

However, the $\mathbf{E}(\mathbf{r})$ in the nth layer is excited by the dipole source locating at $\mathbf{r_s}$. So, we have

$$\mathbf{E}(\mathbf{r}) = \overline{\overline{\mathbf{G}}}_{\mathbf{EJ}}(\mathbf{r},\mathbf{r_s})\cdot\mathbf{I}\delta(\mathbf{r}-\mathbf{r_s}). \tag{8}$$

Based on (6) and (8), (7) can be further derived

$$\begin{aligned}\boldsymbol{\delta}\mathbf{E}(\mathbf{r_r}) = &\frac{j\omega\varepsilon_0}{16\pi^4}\int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dxdy\iint_{-\infty}^{+\infty}dk_xdk_y \\ &\times\tilde{\mathbf{G}}_{\mathbf{EJ}}e^{-j[k_x(x_r-x)+k_y(y_r-y)]}\cdot\boldsymbol{\delta}\overline{\overline{\epsilon}}_r\iint_{-\infty}^{+\infty}dk'_xdk'_y \\ &\times\tilde{\mathbf{G}}_{\mathbf{EJ}}e^{-j\left[k'_x(x-x_s)+k'_y(y-y_s)\right]}\cdot\mathbf{I}\delta(\mathbf{r}-\mathbf{r_s}).\end{aligned} \tag{9}$$

Using identity (A1) given in the Appendix and invoking the diagonal property of $\boldsymbol{\delta}\overline{\overline{\epsilon}}_r$, we can obtain

$$\begin{aligned}\boldsymbol{\delta}\mathbf{E}(\mathbf{r_r}) = &\frac{j\omega\varepsilon_0}{4\pi^2}\int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dk_xdk_ye^{-j[k_x(x_r-x_s)+k_y(y_r-y_s)]} \\ &\times\tilde{\mathbf{G}}_{\mathbf{EJ}}(\mathbf{r_r},\mathbf{r})\cdot(\tilde{\mathbf{G}}_{\mathbf{EJ}}(\mathbf{r},\mathbf{r_s})\cdot\mathbf{I}\delta(\mathbf{r}-\mathbf{r_s}))\circ[\delta\epsilon_h\ \delta\epsilon_h\ \delta\epsilon_v]^T\end{aligned} \tag{10}$$

where $\circ$ represents the Hadamard product and T denotes the matrix transpose. Following the similar procedure, we obtain the magnetic field perturbation at $\mathbf{r_r}$:

$$\begin{aligned}\boldsymbol{\delta}\mathbf{H}(\mathbf{r_r}) = &\frac{j\omega\varepsilon_0}{4\pi^2}\int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dk_xdk_ye^{-j[k_x(x_r-x_s)+k_y(y_r-y_s)]} \\ &\times\tilde{\mathbf{G}}_{\mathbf{HJ}}(\mathbf{r_r},\mathbf{r})\cdot(\tilde{\mathbf{G}}_{\mathbf{EJ}}(\mathbf{r},\mathbf{r_s})\cdot\mathbf{I}\delta(\mathbf{r}-\mathbf{r_s}))\circ[\delta\epsilon_h\ \delta\epsilon_h\ \delta\epsilon_v]^T\end{aligned} \tag{11}$$

where $\tilde{\mathbf{G}}_{\mathbf{EJ}}$ and $\tilde{\mathbf{G}}_{\mathbf{HJ}}$ are derived using the transmission-line network analogy, and their detailed mathematical expressions are given in (28) and (29) of [9]. Now, use the formula given in (A2) and the integral identity given in (A3) of the Appendix, and assume the electric dipole source locating at $\mathbf{r_s}$ is polarized in $\hat{x}$-, $\hat{y}$-, and $\hat{z}$-directions, respectively. Based on (10) and (11), we can derive the elements of the sensitivity matrices in (5). If the electric dipole is $\hat{x}$ polarized, they are as follows:

$$F_{Jx,h}^{ex} = \Im\left\{V_i^eV_i^eF_1 + V_i^hV_i^hF_2\right\} \tag{12a}$$

$$F_{Jx,v}^{ex} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{V_v^eI_i^e\left(J_0\cos^2\phi - \frac{J_1}{k_\rho\rho}\cos2\phi\right)k_\rho^3\right\} \tag{12b}$$

$$F_{Jx,h}^{ey} = \frac{1}{2}\Im\left\{\left(V_i^eV_i^e - V_i^hV_i^h\right)\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin2\phi k_\rho\right\} \tag{12c}$$

$$F_{Jx,v}^{ey} = \frac{1}{2\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{V_v^eI_i^e\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin2\phi k_\rho^3\right\} \tag{12d}$$

$$F_{Jx,h}^{ez} = \frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_r})}\Im\left\{I_i^eV_i^e(jJ_1)\cos\phi k_\rho^2\right\} \tag{12e}$$

$$F_{Jx,v}^{ez} = \frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_r})}\Im\left\{I_v^eI_i^e(jJ_1)\cos\phi k_\rho^4\right\} \tag{12f}$$

$$F_{Jx,h}^{hx} = \frac{1}{2}\Im\left\{\left(I_i^hV_i^h - I_i^eV_i^e\right)\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin2\phi k_\rho\right\} \tag{12g}$$

$$F_{Jx,v}^{hx} = -\frac{1}{2\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{I_v^eI_i^e\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin2\phi k_\rho^3\right\} \tag{12h}$$

$$F_{Jx,h}^{hy} = \Im\left\{I_i^eV_i^eF_1 + I_i^hV_i^hF_2\right\} \tag{12i}$$

$$F_{Jx,v}^{hy} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{I_v^eI_i^e\left(J_0\cos^2\phi - \frac{J_1}{k_\rho\rho}\cos2\phi\right)k_\rho^3\right\} \tag{12j}$$

$$F_{Jx,h}^{hz} = \frac{1}{\omega\mu_0\mu_v(\mathbf{r_r})}\Im\left\{V_i^hV_i^h(jJ_1)\sin\phi k_\rho^2\right\} \tag{12k}$$

where $F_1 = (J_0\cos^2\phi - (J_1/k_\rho\rho)\cos2\phi)k_\rho$ and $F_2 = (J_0\sin^2\phi + (J_1/k_\rho\rho)\cos2\phi)k_\rho$. If the electric dipole is $\hat{y}$ polarized, they are as follows:

$$F^{ex}_{Jy,h} = F^{ey}_{Jx,h} \tag{13a}$$

$$F^{ex}_{Jy,v} = F^{ey}_{Jx,v} \tag{13b}$$

$$F^{ey}_{Jy,h} = \Im\{V_i^h V_i^h F_1 + V_i^e V_i^e F_2\} \tag{13c}$$

$$F^{ey}_{Jy,v} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{V_v^e I_i^e\left(J_0\sin^2\phi + \frac{J_1}{k_\rho\rho}\cos 2\phi\right)k_\rho^3\right\} \tag{13d}$$

$$F^{ez}_{Jy,h} = \frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_r})}\Im\{I_i^e V_i^e(jJ_1)\sin\phi k_\rho^2\} \tag{13e}$$

$$F^{ez}_{Jy,v} = \frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_r})}\Im\{I_v^e I_i^e(jJ_1)\sin\phi k_\rho^4\} \tag{13f}$$

$$F^{hx}_{Jy,h} = \Im\{I_i^h V_i^h F_1 + I_i^e V_i^e F_2\} \tag{13g}$$

$$F^{hx}_{Jy,v} = -\frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{I_v^e I_i^e\left(J_0\sin^2\phi + \frac{J_1}{k_\rho\rho}\cos 2\phi\right)k_\rho^3\right\} \tag{13h}$$

$$F^{hy}_{Jy,h} = -F^{hx}_{Jx,h} \tag{13i}$$

$$F^{hy}_{Jy,v} = -F^{hx}_{Jx,v} \tag{13j}$$

$$F^{hz}_{Jy,h} = -\frac{1}{\omega\mu_0\mu_v(\mathbf{r_r})}\Im\{V_i^h V_i^h(jJ_1)\cos\phi k_\rho^2\}. \tag{13k}$$

If the electric dipole is $\hat{z}$ polarized, they are as follows:

$$F^{ex}_{Jz,h} = \frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_s})}\Im\{V_i^e V_v^e(jJ_1)\cos\phi k_\rho^2\} \tag{14a}$$

$$F^{ex}_{Jz,v} = \frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_s})}\Im\{V_v^e I_v^e(jJ_1)\cos\phi k_\rho^4\} \tag{14b}$$

$$F^{ey}_{Jz,h} = \frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_s})}\Im\{V_i^e V_v^e(jJ_1)\sin\phi k_\rho^2\} \tag{14c}$$

$$F^{ey}_{Jz,v} = \frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_s})}\Im\{V_v^e I_v^e(jJ_1)\sin\phi k_\rho^4\} \tag{14d}$$

$$F^{ez}_{Jz,h} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v(\mathbf{r_r})\epsilon_v(\mathbf{r_s})}\Im\{I_i^e V_v^e(J_0)k_\rho^3\} \tag{14e}$$

$$F^{ez}_{Jz,v} = \frac{1}{\omega^4\varepsilon_0^4\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_r})\epsilon_v(\mathbf{r_s})}\Im\{I_v^e I_v^e(J_0)k_\rho^5\} \tag{14f}$$

$$F^{hx}_{Jz,h} = -\frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_s})}\Im\{I_i^e V_v^e(jJ_1)\sin\phi k_\rho^2\} \tag{14g}$$

$$F^{hx}_{Jz,v} = -\frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_s})}\Im\{I_v^e I_v^e(jJ_1)\sin\phi k_\rho^4\} \tag{14h}$$

$$F^{hy}_{Jz,h} = \frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_s})}\Im\{I_i^e V_v^e(jJ_1)\cos\phi k_\rho^2\} \tag{14i}$$

$$F^{hy}_{Jz,v} = \frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_s})}\Im\{I_v^e I_v^e(jJ_1)\cos\phi k_\rho^4\}. \tag{14j}$$

In the above (12)–(14), $F^{hz}_{Jx,v}$, $F^{hz}_{Jy,v}$, $F^{hz}_{Jz,h}$, and $F^{hz}_{Jz,v}$ are not listed because they are equal to zero. The expression $\Im\{\} = (1/2\pi)\int_{z_{n-1}}^{z_n}\int_0^{+\infty}\{\}dk_\rho dz$ denotes the double integral, $J_0 = J_0(k_\rho\rho)$ and $J_1 = J_1(k_\rho\rho)$ are the zeroth- and first-order Bessel functions, respectively, with the argument $k_\rho\rho$, $\rho = \sqrt{(x_r - x_s)^2 + (y_r - y_s)^2}$ is the horizontal distance between the dipole source and the receiver, and $\epsilon_v(\mathbf{r_s})$, $\epsilon_v(\mathbf{r})$, and $\epsilon_v(\mathbf{r_r})$ are the ϵ_v values at the positions of $\mathbf{r_s}$, $\mathbf{r}$, and $\mathbf{r_r}$, respectively. The first term of the multiplication of the voltage and current terms is determined by z_r and z, while the second term is determined by z and z_s. For example, in (12b), $V_v^e I_i^e = V_v^e(z_r, z)I_i^e(z, z_s)$. The superscript e stands for the TM mode, while h stands for the TE mode. $V_v^e(z_r, z)$ represents the TM component of the spectral domain transverse electric field and is denoted by the voltage V measured at z_r excited by 1 V series voltage source v locating at z. $I_i^e(z, z_s)$ represents the TM component of the spectral domain transverse magnetic field and is denoted by the current I measured at z excited by a 1 A shunt current source i locating at z_s. They are shown in Fig. 4 of [9] and their evaluation can be found in (62)–(70) of [9]. The analogy between the true dipole sources and the voltage source v and the current source i is shown in Fig. 2 of [9], and their mathematical relationships are given in (19) of [9]. One should note that the order of the arguments of the voltage or current cannot be directly interchanged. The reciprocity theorem must be applied, which will be discussed in Section IV.

IV. Sensitivity Matrices Due to a Magnetic Dipole

If the layered TI medium is illuminated by the EM wave excited by a magnetic dipole source $\mathbf{M} = \mathbf{\Psi}\boldsymbol{\delta}(\mathbf{r} - \mathbf{r_s})$, we can follow the similar procedure in (8)–(11) and come to the field perturbations:

$$\begin{aligned}\delta\mathbf{E}(\mathbf{r_r}) = &\frac{j\omega\varepsilon_0}{4\pi^2}\int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dk_x dk_y e^{-j[k_x(x_r-x_s)+k_y(y_r-y_s)]}\\ &\times\tilde{\mathbf{G}}_{\mathbf{EJ}}(\mathbf{r_r},\mathbf{r})\cdot(\tilde{\mathbf{G}}_{\mathbf{EM}}(\mathbf{r},\mathbf{r_s})\cdot\mathbf{\Psi}\delta(\mathbf{r}-\mathbf{r_s}))\\ &\circ[\delta\epsilon_h\ \delta\epsilon_h\ \delta\epsilon_v]^T\end{aligned} \tag{15}$$

$$\begin{aligned}\delta\mathbf{H}(\mathbf{r_r}) = &\frac{j\omega\varepsilon_0}{4\pi^2}\int_{z_{n-1}}^{z_n}dz\iint_{-\infty}^{+\infty}dk_x dk_y e^{-j[k_x(x_r-x_s)+k_y(y_r-y_s)]}\\ &\times\tilde{\mathbf{G}}_{\mathbf{HJ}}(\mathbf{r_r},\mathbf{r})\cdot(\tilde{\mathbf{G}}_{\mathbf{EM}}(\mathbf{r},\mathbf{r_s})\cdot\mathbf{\Psi}\delta(\mathbf{r}-\mathbf{r_s}))\\ &\circ[\delta\epsilon_h\ \delta\epsilon_h\ \delta\epsilon_v]^T.\end{aligned} \tag{16}$$

Then, the elements of the sensitivity matrices in (5) can be found similarly. If the magnetic dipole is $\hat{x}$ polarized, they are as follows:

$$F^{ex}_{Mx,h} = \frac{1}{2}\Im\left\{\left(V_i^h V_v^h - V_i^e V_v^e\right)\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin 2\phi k_\rho\right\} \tag{17a}$$

$$F^{ex}_{Mx,v} = -\frac{1}{2\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{V_v^e I_v^e\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin 2\phi k_\rho^3\right\} \tag{17b}$$

$$F^{ey}_{Mx,h} = -\Im\{V_i^h V_v^h F_1 + V_i^e V_v^e F_2\} \tag{17c}$$

$$F^{ey}_{Mx,v} = -\frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{V_v^e I_v^e\left(J_0\sin^2\phi + \frac{J_1}{k_\rho\rho}\cos 2\phi\right)k_\rho^3\right\} \tag{17d}$$

$$F^{ez}_{Mx,h} = -\frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_r})}\Im\{I_i^e V_v^e(jJ_1)\sin\phi k_\rho^2\} \tag{17e}$$

$$F^{ez}_{Mx,v} = -\frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_r})}\Im\{I_v^e I_v^e(jJ_1)\sin\phi k_\rho^4\} \tag{17f}$$

$$F^{hx}_{Mx,h} = \Im\{I_i^h V_v^h F_1 + I_i^e V_v^e F_2\} \tag{17g}$$

$$F^{hx}_{Mx,v} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{I_v^e I_v^e\left(J_0\sin^2\phi + \frac{J_1}{k_\rho\rho}\cos 2\phi\right)k_\rho^3\right\} \tag{17h}$$

$$F^{hy}_{Mx,h} = \frac{1}{2}\Im\left\{\left(I_i^h V_v^h - I_i^e V_v^e\right)\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin 2\phi k_\rho\right\} \tag{17i}$$

$$F^{hy}_{Mx,v} = -\frac{1}{2\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{I_v^e I_v^e\left(J_0 - \frac{2J_1}{k_\rho\rho}\right)\sin 2\phi k_\rho^3\right\} \tag{17j}$$

$$F^{hz}_{Mx,h} = \frac{1}{\omega\mu_0\mu_v(\mathbf{r_r})}\Im\{V_i^h V_v^h(jJ_1)\cos\phi k_\rho^2\}. \tag{17k}$$

If the magnetic dipole is $\hat{y}$ polarized, they are as follows:

$$F^{ex}_{My,h} = \Im\{V_i^e V_v^e F_1 + V_i^h V_v^h F_2\} \tag{18a}$$

$$F^{ex}_{My,v} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{V_v^e I_v^e\left(J_0\cos^2\phi - \frac{J_1}{k_\rho\rho}\cos 2\phi\right)k_\rho^3\right\} \tag{18b}$$

$$F^{ey}_{My,h} = -F^{ex}_{Mx,h} \tag{18c}$$

$$F^{ey}_{My,v} = -F^{ex}_{Mx,v} \tag{18d}$$

$$F^{ez}_{My,h} = \frac{1}{\omega\varepsilon_0\epsilon_v(\mathbf{r_r})}\Im\{I_i^e V_v^e(jJ_1)\cos\phi k_\rho^2\} \tag{18e}$$

$$F^{ez}_{My,v} = \frac{1}{\omega^3\varepsilon_0^3\epsilon_v^2(\mathbf{r})\epsilon_v(\mathbf{r_r})}\Im\{I_v^e I_v^e(jJ_1)\cos\phi k_\rho^4\} \tag{18f}$$

$$F^{hx}_{My,h} = F^{hy}_{Mx,h} \tag{18g}$$
$$F^{hx}_{My,v} = F^{hy}_{Mx,v} \tag{18h}$$
$$F^{hy}_{My,h} = \Im\{I^e_i V^e_v F_1 + I^h_i V^h_v F_2\} \tag{18i}$$
$$F^{hy}_{My,v} = \frac{1}{\omega^2\varepsilon_0^2\epsilon_v^2(\mathbf{r})}\Im\left\{I^e_v I^e_v\left(J_0\cos^2\phi - \frac{J_1}{k_\rho \rho}\cos 2\phi\right)k_\rho^3\right\} \tag{18j}$$
$$F^{hz}_{My,h} = \frac{1}{\omega\mu_0\mu_v(\mathbf{r_r})}\Im\{V^h_i V^h_v (jJ_1)\sin\phi k_\rho^2\}. \tag{18k}$$

If the magnetic dipole is $\hat{z}$ polarized, they are as follows:

$$F^{ex}_{Mz,h} = \frac{1}{\omega\mu_0\mu_v(\mathbf{r_s})}\Im\{V^h_i V^h_i (jJ_1)\sin\phi k_\rho^2\} \tag{19a}$$
$$F^{ey}_{Mz,h} = -\frac{1}{\omega\mu_0\mu_v(\mathbf{r_s})}\Im\{V^h_i V^h_i (jJ_1)\cos\phi k_\rho^2\} \tag{19b}$$
$$F^{hx}_{Mz,h} = \frac{1}{\omega\mu_0\mu_v(\mathbf{r_s})}\Im\{I^h_i V^h_i (jJ_1)\cos\phi k_\rho^2\} \tag{19c}$$
$$F^{hy}_{Mz,h} = \frac{1}{\omega\mu_0\mu_v(\mathbf{r_s})}\Im\{I^h_i V^h_i (jJ_1)\sin\phi k_\rho^2\} \tag{19d}$$
$$F^{hz}_{Mz,h} = \frac{1}{\omega^2\mu_0^2\mu_v(\mathbf{r_r})\mu_v(\mathbf{r_s})}\Im\{V^h_i V^h_i (J_0)k_\rho^3\}. \tag{19e}$$

In the above (17)–(19), $F^{hz}_{Mx,v}$, $F^{hz}_{My,v}$, $F^{ex}_{Mz,v}$, $F^{ey}_{Mz,v}$, $F^{ez}_{Mz,h}$, $F^{ez}_{Mz,v}$, $F^{hx}_{Mz,v}$, $F^{hy}_{Mz,v}$, and $F^{hz}_{Mz,v}$ are not listed because they are equal to zero.

V. Analytical Solutions for Integrals of dz

In the above (12)–(14) and (17)–(19), $\int_{z_{n-1}}^{z_n}\{\}dz$ must be evaluated before $\int_0^{+\infty}\{\}dk_\rho$ and can come to analytical solutions. Since the integrand of $\int_{z_{n-1}}^{z_n}\{\}dz$ is the multiplication of voltage and current terms, we first use the reciprocity formula (24) of [9] to interchange the arguments of the first term in the multiplication. For example, in (12b), the integral of dz can be modified as

$$\int_{z_{n-1}}^{z_n} V^e_v(z_r,z)I^e_i(z,z_s)dz = -\int_{z_{n-1}}^{z_n} I^e_i(z,z_r)I^e_i(z,z_s)dz. \tag{20}$$

In this way, the voltage or current in both terms of the integrand is measured at z excited by the source locating at z_r and z_s, respectively. Such an equivalent transformation can facilitate the following derivation. For brevity, we use z' to denote the source locating at z_s or z_r. When both z and z' locate inside the nth layer, the voltage or current measured at z excited by the source at z' can be evaluated by [9]

$$\begin{cases} V^p_i(z) \\ \quad = \dfrac{Z^p_n}{2}\left[E^p_{0n} + \dfrac{1}{D^p_n}\left(\overrightarrow{\Gamma}^p_n E^p_{1n} + \overleftarrow{\Gamma}^p_n E^p_{2n} + \overrightarrow{\Gamma}^p_n\overleftarrow{\Gamma}^p_n\left(E^p_{3n}+E^p_{4n}\right)\right)\right] \\ V^p_v(z) \\ \quad = \dfrac{1}{2}\left[\pm E^p_{0n} + \dfrac{1}{D^p_n}\left(\overrightarrow{\Gamma}^p_n E^p_{1n} - \overleftarrow{\Gamma}^p_n E^p_{2n} + \overrightarrow{\Gamma}^p_n\overleftarrow{\Gamma}^p_n\left(E^p_{3n}-E^p_{4n}\right)\right)\right] \\ I^p_i(z) \\ \quad = \dfrac{1}{2}\left[\pm E^p_{0n} - \dfrac{1}{D^p_n}\left(\overrightarrow{\Gamma}^p_n E^p_{1n} - \overleftarrow{\Gamma}^p_n E^p_{2n} - \overrightarrow{\Gamma}^p_n\overleftarrow{\Gamma}^p_n\left(E^p_{3n}-E^p_{4n}\right)\right)\right] \\ I^p_v(z) \\ \quad = \dfrac{Y^p_n}{2}\left[E^p_{0n} - \dfrac{1}{D^p_n}\left(\overrightarrow{\Gamma}^p_n E^p_{1n} + \overleftarrow{\Gamma}^p_n E^p_{2n} - \overrightarrow{\Gamma}^p_n\overleftarrow{\Gamma}^p_n\left(E^p_{3n}+E^p_{4n}\right)\right)\right] \end{cases} \tag{21}$$

where the superscript p can be e or h which stands for the TM and TE wave mode, respectively. $\pm$ takes $+$ when $z > z'$ but $-$ when $z < z'$. Z^p_n and Y^p_n denote the characteristic impedance and admittance of the nth layer, respectively, whose expressions are given in (17) and (18) of [9]. $E^p_{0n} = e^{-jk^p_{zn}|z-z'|}$ is the primary field. Other coefficients and variables are evaluated using $D^p_n = 1 - \overrightarrow{\Gamma}^p_n\overleftarrow{\Gamma}^p_n e^{-j2k^p_{zn}d_n}$, $E^p_{1n} = e^{-jk^p_{zn}(d^{hf}_n + d^{hs}_n)}$, $E^p_{2n} = e^{-jk^p_{zn}(d^{lf}_n + d^{ls}_n)}$, $E^p_{3n} = e^{-jk^p_{zn}(d_n + d^{hs}_n + d^{lf}_n)}$, and $E^p_{4n} = e^{-jk^p_{zn}(d_n + d^{hf}_n + d^{ls}_n)}$ where $d_n = z_n - z_{n-1}$, $d^{hf}_n = z_n - z$, $d^{hs}_n = z_n - z'$, $d^{lf}_n = z - z_{n-1}$, and $d^{ls}_n = z' - z_{n-1}$. The global reflection coefficients $\overleftarrow{\Gamma}^p_n$ and $\overrightarrow{\Gamma}^p_n$ are evaluated using the recursive expressions

$$\begin{cases} \overleftarrow{\Gamma}^p_n = \left(\Gamma^p_{n-1,n} + \overleftarrow{\Gamma}^p_{n-1}\, t^p_{n-1}\right)/\left(1 + \Gamma^p_{n-1,n} + \overleftarrow{\Gamma}^p_{n-1}\, t^p_{n-1}\right) \\ \overrightarrow{\Gamma}^p_n = \left(\Gamma^p_{n+1,n} + \overrightarrow{\Gamma}^p_{n+1}\, t^p_{n+1}\right)/\left(1 + \Gamma^p_{n+1,n} + \overrightarrow{\Gamma}^p_{n+1}\, t^p_{n+1}\right) \end{cases} \tag{22}$$

where $\Gamma^p_{i,j} = (Z^p_i - Z^p_j)/(Z^p_i + Z^p_j)$ is the Fresnel reflection coefficient and $t^p_n = e^{-j2k^p_{zn}d_n}$. When z' and z are not in the same layer, we assume z' is in the mth layer and z is in the nth layer. The voltage and current at z can be computed by

$$\begin{cases} V^p(z) = V^p(z_m)\dfrac{\prod_{i=m+1}^{n-1}\dfrac{\left(1+\overrightarrow{\Gamma}^p_i\right)e^{-jk^p_{zi}d_i}}{1+\overrightarrow{\Gamma}^p_i t^p_i}}{\left(1+\overrightarrow{\Gamma}^p_n t^p_n\right)e^{jk^p_{zn}d^{lf}_n}}\left(1+\overrightarrow{\Gamma}^p_n e^{-j2k^p_{zn}d^{hf}_n}\right) \\ I^p(z) = -\dfrac{V^p(z_m)}{Z^p_m}\dfrac{\prod_{i=m+1}^{n-1}\dfrac{\left(1+\overrightarrow{\Gamma}^p_i\right)e^{-jk^p_{zi}d_i}}{1+\overrightarrow{\Gamma}^p_i t^p_i}}{\left(1+\overrightarrow{\Gamma}^p_n t^p_n\right)e^{jk^p_{zn}d^{lf}_n}}\left(1-\overrightarrow{\Gamma}^p_n e^{-j2k^p_{zn}d^{hf}_n}\right) \end{cases} \tag{23}$$

if $m < n$, and

$$\begin{cases} V^p(z) = V^p(z_{m-1})\dfrac{\prod_{i=n}^{m-2}\dfrac{\left(1+\overleftarrow{\Gamma}^p_i\right)e^{-jk^p_{zi}d_i}}{1+\overleftarrow{\Gamma}^p_i t^p_i}}{\left(1+\overleftarrow{\Gamma}^p_n t^p_n\right)e^{jk^p_{zn}d^{hf}_n}}\left(1+\overleftarrow{\Gamma}^p_n e^{-j2k^p_{zn}d^{lf}_n}\right) \\ I^p(z) = -\dfrac{V^p(z_{m-1})}{Z^p_m}\dfrac{\prod_{i=n}^{m-2}\dfrac{\left(1+\overleftarrow{\Gamma}^p_i\right)e^{-jk^p_{zi}d_i}}{1+\overleftarrow{\Gamma}^p_i t^p_i}}{\left(1+\overleftarrow{\Gamma}^p_n t^p_n\right)e^{jk^p_{zn}d^{hf}_n}}\left(1-\overleftarrow{\Gamma}^p_n e^{-j2k^p_{zn}d^{lf}_n}\right) \end{cases} \tag{24}$$

if $m > n$. One should note that some variables in (23) and (24) with the subscript i can be directly found by replacing n with i for the corresponding variables in (21) and (22). $V^p(z_m)$ and $V^p(z_{m-1})$ are evaluated using (21). Once V_p and I^p are obtained, we can derive the expressions of $\int_{z_{n-1}}^{z_n}\{\}dz$ analytically. For convenience, the terms in (21), (23), and (24) uncorrelated to z are discarded in the following derivations. In other words, $(Z^p_n/2)$, $(1/2)$, and $(Y^p_n/2)$ in (21) are discarded. Only $(1/(e^{jk^p_{zn}d^{lf}_n})(1 \pm \overrightarrow{\Gamma}^p_n e^{-j2k^p_{zn}d^{hf}_n})$ in (23) and only $(1/(e^{jk^p_{zn}d^{hf}_n})(1 \pm \overrightarrow{\Gamma}^p_n e^{-j2k^p_{zn}d^{lf}_n})$ in (24) are kept in the following derivations for $\int_{z_{n-1}}^{z_n}\{\}dz$. In addition, it is not allowed to exchange two terms of the integrand since the first one has the argument of (z, z_r) and the second one has the argument of (z, z_s). Meanwhile, the sign generated by the reciprocity transform, for example, the negative sign in the right side of (20), is also discarded in the following derivations. We assume that z_r locates in the m_rth layer and z_s locates in the m_sth layer. Then, it is straightforward to obtain

$$\int_{z_{n-1}}^{z_n}\{\}dz = \frac{\left(t^p_n - 1\right)}{-j2k^p_{zn}}\left(1 + \overrightarrow{\Gamma}^p_n\overrightarrow{\Gamma}^p_n\, t^p_n\right) \pm 2\,\overrightarrow{\Gamma}^p_n\, t^p_n d_n \tag{25}$$

if $m_r < n$ and $m_s < n$, and

$$\int_{z_{n-1}}^{z_n}\{\}dz = \frac{\left(t^p_n - 1\right)}{-j2k^p_{zn}}\left(1 + \overleftarrow{\Gamma}^p_n\overleftarrow{\Gamma}^p_n\, t^p_n\right) \pm 2\,\overleftarrow{\Gamma}^p_n\, t^p_n d_n \tag{26}$$

if $m_r > n$ and $m_s > n$. When the dipole source and the receiver locate in two sides of the nth layer in which z locates, that is, $m_r > n > m_s$

or $m_r < n < m_s$, the integration of dz from z_{n-1} to z_n is

$$e^{-jk_{zn}^p d_n} d_n\left(1+ \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p t_n^p\right) \pm \frac{e^{-jk_{zn}^p d_n}}{-j2k_{zn}^p}\left(\overleftarrow{\Gamma}_n^p + \overrightarrow{\Gamma}_n^p \right)\left(t_n^p - 1\right) \tag{27}$$

where $\pm$ takes $+$ for the integrand of $V^p V^p$ but $-$ for the integrand of $I^p I^p$. It is worth mentioning that two terms of the integrands in (12)–(14) and (17)–(19) for dz only take $V^p V^p$ or $I^p I^p$ after the transforms similar to that used in (20). There is no $V^p I^p$ term.

If the dipole source locates inside the nth layer ($n = m_s$) and the receiver locates outside the nth layer or the receiver locates inside the nth layer ($n = m_r$) and the dipole source locates outside the nth layer, we can obtain $\int_{z_{n-1}}^{z_n}\{\}dz$ by taking the integration of the multiplication of (21) and (23) or (21) and (24) from z_{n-1} to z_n. The result is

$$\begin{aligned}
&SA_1^p + \frac{1}{-j2k_{zn}^p}A_2^p \pm \overrightarrow{\Gamma}_n^p A_3^p \pm \frac{\overrightarrow{\Gamma}_n^p}{-j2k_{zn}^p} SA_4^p \\
&\pm \frac{1}{D_n^p}\left\{ \overrightarrow{\Gamma}_n^p B_1^p \pm \frac{\overrightarrow{\Gamma}_n^p\left(\overleftarrow{\Gamma}_n^p + \overrightarrow{\Gamma}_n^p \right)}{-j2k_{zn}^p} B_2^p + \frac{\overleftarrow{\Gamma}_n^p}{-j2k_{zn}^p} SB_3^p \right. \\
&\qquad \left. \pm 2\, \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p SB_4^p + \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p B_5^p + \frac{\overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p}{-j2k_{zn}^p} SB_6^p \right\}
\end{aligned} \tag{28}$$

if $n > m_r$ or $n > m_s$, and

$$\begin{aligned}
&\frac{1}{-j2k_{zn}^p} SC_1^p + C_2^p \pm \frac{\overleftarrow{\Gamma}_n^p}{-j2k_{zn}^p} C_3^p \pm \overleftarrow{\Gamma}_n^p SC_4^p \\
&\pm \frac{1}{D_n^p}\left\{ \frac{\overrightarrow{\Gamma}_n^p}{-j2k_{zn}^p} D_1^p \pm 2\, \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p D_2^p + \overleftarrow{\Gamma}_n^p SD_3^p \right. \\
&\qquad \pm \frac{\overleftarrow{\Gamma}_n^p\left(\overleftarrow{\Gamma}_n^p + \overrightarrow{\Gamma}_n^p \right)}{-j2k_{zn}^p} SD_4^p + \frac{\overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \overleftarrow{\Gamma}_n^p}{-j2k_{zn}^p} D_5^p \\
&\qquad \left. + \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \overleftarrow{\Gamma}_n^p SD_6^p \right\}
\end{aligned} \tag{29}$$

if $n < m_r$ or $n < m_s$, where

$$\begin{cases}
A_1^p = e^{-jk_{zn}^p d_n^{ls}} d_n^{ls}, \quad C_1^p = e^{-jk_{zn}^p\left(d_n + d_n^{ls}\right)} - e^{-jk_{zn}^p d_n^{hs}} \\
A_2^p = e^{-jk_{zn}^p\left(d_n + d_n^{hs}\right)} - e^{-jk_{zn}^p d_n^{ls}}, \quad C_2^p = e^{-jk_{zn}^p d_n^{hs}} d_n^{hs} \\
A_3^p = e^{-jk_{zn}^p(d_n + d_n^{hs})} d_n^{hs}, \quad C_3^p = t_n^p e^{-jk_{zn}^p d_n^{hs}} - e^{-jk_{zn}^p\left(d_n + d_n^{ls}\right)} \\
A_4^p = t_n^p e^{-jk_{zn}^p d_n^{ls}} - e^{-jk_{zn}^p\left(d_n + d_n^{hs}\right)}, \quad C_4^p = e^{-jk_{zn}^p\left(d_n + d_n^{ls}\right)} d_n^{ls} \\
B_1^p = e^{-jk_{zn}^p\left(d_n + d_n^{hs}\right)} d_n, \quad D_1^p = \left(t_n^p - 1\right) e^{-jk_{zn}^p d_n^{hs}} \\
B_2^p = \left(t_n^p - 1\right) e^{-jk_{zn}^p\left(d_n + d_n^{hs}\right)}, \quad D_2^p = t_n^p e^{-jk_{zn}^p d_n^{hs}} d_n \\
B_3^p = \left(t_n^p - 1\right) e^{-jk_{zn}^p d_n^{ls}}, \quad D_3^p = e^{-jk_{zn}^p\left(d_n + d_n^{ls}\right)} d_n \\
B_4^p = t_n^p e^{-jk_{zn}^p d_n^{ls}} d_n, \quad D_4^p = \left(t_n^p - 1\right) e^{-jk_{zn}^p\left(d_n + d_n^{ls}\right)} \\
B_5^p = t_n^p e^{-jk_{zn}^p\left(d_n + d_n^{hs}\right)} d_n, \quad D_5^p = t_n^p\left(t_n^p - 1\right) e^{-jk_{zn}^p d_n^{hs}} \\
B_6^p = t_n^p\left(t_n^p - 1\right) e^{-jk_{zn}^p d_n^{ls}}, \quad D_6^p = t_n^p e^{-jk_{zn}^p\left(d_n + d_n^{ls}\right)} d_n \\
d_n^{ls} = z_s - z_{n-1}, \quad d_n^{hs} = z_n - z_s, \quad \text{if } n = m_s \\
d_n^{ls} = z_r - z_{n-1}, \quad d_n^{hs} = z_n - z_r, \quad \text{if } n = m_r.
\end{cases} \tag{30}$$

In (28) and (29), $\pm$ takes $+$ for the integrand of $V^p V^p$ but $-$ for $I^p I^p$. When the source locates inside the nth layer, that is, $n = m_s$, S is -1 for the integrands of $V_i^p V_v^p$, $V_v^p V_v^p$, $I_i^p I_i^p$, and $I_v^p I_i^p$ but 1 for the integrands of $V_i^p V_i^p$, $V_v^p V_i^p$, $I_i^p I_v^p$, and $I_v^p I_v^p$. However, when the receiver locates inside the nth layer, that is, $n = m_r$, S is -1 for the integrands of $V_v^p V_v^p$, $V_v^p V_i^p$, $I_i^p I_v^p$, and $I_i^p I_i^p$ but 1 for the integrands of $V_i^p V_v^p$, $V_i^p V_i^p$, $I_v^p I_v^p$, and $I_v^p I_i^p$.

If both the dipole source and the receiver locate in the nth layer, that is, $n = m_s = m_r$, $\int_{z_{n-1}}^{z_n}\{\}dz$ is calculated by taking the integration of the multiplication of two terms in (21) from z_{n-1} to z_n. It is

$$\begin{aligned}
&\frac{1}{-j2k_{zn}^p}\left\{S_1 Q_1^p + Q_2^p\right\} + S_5 Q_5^p + \frac{S_0}{-j2k_{zn}^p} \\
&\times\left\{-2Q_6^p + \frac{2S_2}{D_n^p}\left(\overleftarrow{\Gamma}_n^p Q_1^p + \overrightarrow{\Gamma}_n^p Q_2^p\right)\right\} + \left(\frac{1}{D_n^p}\right)^2 \frac{1}{-j2k_{zn}^p} \\
&\times\left\{\left(\overleftarrow{\Gamma}_n^p \right)^2\left(t_n^p - 1\right) S_1 Q_1^p + \left(\overrightarrow{\Gamma}_n^p \right)^2\left(t_n^p - 1\right) Q_2^p\right\} \\
&+ \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \left(1 + \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p t_n^p\right) e^{-jk_{zn}^p d_n} d_n \left(\frac{1}{D_n^p}\right)^2 \\
&\times\left\{S_3 Q_3^p + S_4 Q_4^p\right\} \pm 2\left(\frac{1}{D_n^p}\right)^2 \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p t_n^p d_n \\
&\times\left(\overleftarrow{\Gamma}_n^p S_1 Q_1^p + \overrightarrow{\Gamma}_n^p Q_2^p\right) \pm \frac{1}{D_n^p} \\
&\times\left\{ \overleftarrow{\Gamma}_n^p S_1 Q_1^p P_1 + \overrightarrow{\Gamma}_n^p Q_2^p P_2 \pm \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p e^{-jk_{zn}^p d_n} \right. \\
&\quad \left. \times\left(S_3 Q_3^p P_3 + S_4 Q_4^p P_4\right)\right\} \\
&+\left(\frac{2}{D_n^p} + \left(\frac{1}{D_n^p}\right)^2 \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \left(t_n^p - 1\right)\right) \frac{\overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p t_n^p}{-j2k_{zn}^p}\left\{S_1 Q_1^p + Q_2^p\right\} \\
&\pm\left(\left(\frac{1}{D_n^p}\right)^2 \overleftarrow{\Gamma}_n^p \overrightarrow{\Gamma}_n^p \left(t_n^p - 1\right) + \frac{1}{D_n^p}\right) \\
&\times\left(\overleftarrow{\Gamma}_n^p + \overrightarrow{\Gamma}_n^p \right) \frac{e^{-jk_{zn}^p d_n}}{-j2k_{zn}^p}\left\{S_3 Q_3^p + S_4 Q_4^p\right\}
\end{aligned} \tag{31}$$

where

$$\begin{cases}
P_1 = z_s + z_r - 2z_{n-1}, \quad P_2 = 2z_n - z_s - z_r \\
P_3 = z_s - z_r + d_n, \quad P_4 = z_r - z_s + d_n, \quad Q_1^p = e^{-jk_{zn}^p P_1} \\
Q_2^p = e^{-jk_{zn}^p P_2}, \quad Q_3^p = e^{-jk_{zn}^p P_3}, \quad Q_4^p = e^{-jk_{zn}^p P_4} \\
Q_5^p = e^{-jk_{zn}^p |z_s - z_r|}|z_s - z_r|, \quad Q_6^p = e^{-jk_{zn}^p |z_s - z_r|}.
\end{cases} \tag{32}$$

In (31), $\pm$ takes $+$ for the integrand of $V^p V^p$ but $-$ for $I^p I^p$. S_0 is 0 and S_1 is -1 for the integrands of $V_i^p V_v^p$, $V_v^p V_i^p$, $I_i^p I_v^p$, and $I_v^p I_i^p$. However, both of them become 1 for the integrands of $V_v^p V_v^p$, $V_i^p V_i^p$, $I_v^p I_v^p$, and $I_i^p I_i^p$. S_2 is -1 for the integrands of $V_i^p V_i^p$ and $I_i^p I_i^p$ but 1 for the integrands of $V_v^p V_v^p$ and $I_v^p I_v^p$. S_3 is -1 for the integrands of $V_i^p V_v^p$, $V_v^p V_v^p$, $I_i^p I_i^p$, and $I_v^p I_i^p$ but 1 for the integrands of $V_i^p V_i^p$, $V_v^p V_i^p$, $I_i^p I_v^p$, and $I_v^p I_v^p$. S_4 is -1 for the integrands of $V_v^p V_v^p$, $V_v^p V_i^p$, $I_i^p I_v^p$, and $I_i^p I_i^p$ but 1 for the integrands of $V_i^p V_v^p$, $V_i^p V_i^p$, $I_v^p I_v^p$, and $I_v^p I_i^p$. S_5 is equal to S_3 when $z_r < z_s$. However, it is equal to S_4 when $z_r > z_s$.

VI. Sensitivity Matrices Due to Boundary Changes

In the aforementioned derivations, (12)–(14) and (17)–(19) are only for the perturbation of relative permittivity in the nth layer. However, in many geophysical applications, it is also necessary to reconstruct the layer boundary positions. To find the sensitivity matrices of the measured $\mathbf{E}$ and $\mathbf{H}$ at the receiver with respect to the z_n variation, we assume it has a small perturbation of δz_n. Therefore, a perturbed thin layer forms between z_n and $z_n + \delta z_n$. Compared with the dielectric parameters and field values in this thin layer before the perturbation, the relative permittivity changes $\delta\bar{\bar{\epsilon}}_r = \bar{\bar{\epsilon}}_{r,n} - \bar{\bar{\epsilon}}_{r,n+1}$ after the perturbation. The horizontal component of $\mathbf{E}$-field in the perturbed thin layer remains unchanged. However, the vertical component becomes $((\epsilon_{v,n+1}E_z)/\epsilon_{v,n})$ since the normal component of the flux is continuous across the layer boundary. Therefore, the equivalent electric source $\mathbf{J}'$ in (4b) inside the perturbed thin layer is

$$\mathbf{J}' = j\omega\varepsilon_0 \left\{ (\epsilon_{h,n} - \epsilon_{h,n+1})(\hat{x}E_x + \hat{y}E_y) + \epsilon_{v,n+1} \times \left(\frac{\epsilon_{v,n+1}}{\epsilon_{v,n}} - 1 \right) \hat{z}E_z \right\}. \quad (33)$$

We can see that the field perturbations $\boldsymbol{\delta}\mathbf{E}$ and $\boldsymbol{\delta}\mathbf{H}$ at the receiver caused by δz_n are the same as the those caused by the permittivity change $\delta\epsilon_h = \epsilon_{h,n} - \epsilon_{h,n+1}$ and $\delta\epsilon_v = \epsilon_{v,n+1}((\epsilon_{v,n+1}/\epsilon_{v,n}) - 1)$ inside the perturbed thin layer. We can still use (12)–(14) and (17)–(19) to compute the sensitivity matrices for δz_n. Because the integral of dz is from z_n to $z_n + \delta z_n$ and δz_n is tiny, (20) becomes

$$\int_{z_n}^{z_n+\delta z_n} V_v^e(z_r, z) I_i^e(z, z_s) dz = -\delta z_n I_i^e(z_n^+, z_r) I_i^e(z_n^+, z_s) \quad (34)$$

where z_n^+ is a number a little larger than z_n but smaller than $z_n + \delta z_n$. Consequently, the sensitivity matrices for δz_n are computed by discarding the integral of dz in (12)–(14) and (17)–(19), replacing the variable z in the voltage and current terms with z_n^+, multiplying the horizontal component with $\epsilon_{h,n} - \epsilon_{h,n+1}$ and the vertical component with $\epsilon_{v,n+1}((\epsilon_{v,n+1}/\epsilon_{v,n})-1)$, and finally adding them together. There is no need to use (25)–(32) to evaluate the integration.

VII. Implementation of Numerical Integration

Before using the Gaussian quadratures to compute the Sommerfeld integration, we want to emphasize three points. First, when the source or receiver locates in the first or the last layer or when the perturbed nth layer is the first or the last layer, (25)–(32) can be simplified by placing a fictitious layer boundary in the infinity. Let us take (25) as an example. If we put a fictitious layer boundary z_N very far from z_{N-1}, $\overrightarrow{\Gamma}_n^p$ becomes zero and t_n^p also approaches zero since d_n is very large. Equation (25) degenerates into $1/(j2k_{zn}^p)$. Second, the integration path for k_ρ varying from zero to infinity must be deformed from the real axis to avoid the integrand singularity. See Fig. 4 of [10]. Third, the exponential decaying advantage of the integrand will disappear if the source or the receiver approaches the layer boundary or if they are close to each other. As a result, the integrand is dominated by the oscillatory Bessel functions and the convergence of the Sommerfeld integration will be slow. The subtraction technique in [10] can be used to accelerate the convergence. In a nutshell, we subtract the large-argument asymptotic form of the integrand inside the integration and add the corresponding analytical integration solution later. The added analytical expressions can be found in (33)–(36) of [10].

VIII. Conclusion

In this communication, the semianalytical expressions of sensitivity matrices for the reconstruction of 1-D layered TI media are derived. The 3-D dipole source can be either electric or magnetic and has arbitrary polarization. The obtained mathematical expressions are ready for computer coding and thus play an important role in solving inversion problems of 1-D TI media illuminated by 3-D sources.

Appendix

The following integral formula of the 2-D Dirac's function holds:

$$\iint_{-\infty}^{+\infty} dxdy e^{-j\left[\left(-k_x+k_x'\right)x+\left(-k_y+k_y'\right)y\right]} = 4\pi^2\delta\left(-k_x+k_x'\right)\delta\left(-k_y+k_y'\right). \quad (A1)$$

The following formula transforms the integral of $dk_x dk_y$ into the integral of dk_ρ:

$$\iint_{-\infty}^{+\infty} f(k_x, k_y, k_\rho, z, z_s, z_r) dk_x dk_y e^{-j[k_x(x_r-x_s)+k_y(y_r-y_s)]} = \int_0^{+\infty} dk_\rho \int_0^{2\pi} f(k_x, k_y, k_\rho, z, z_s, z_r) k_\rho e^{-j\boldsymbol{k_\rho}\cdot\boldsymbol{\rho}\cos(\xi-\phi)} d\xi \quad (A2)$$

where ξ is the angle between the vector $\boldsymbol{k_\rho} = \hat{x}k_x + \hat{y}k_y$ and the $\hat{x}$-axis, while ϕ is the angle between the vector $\boldsymbol{\rho} = \hat{x}(x_r - x_s) + \hat{y}(y_r - y_s)$ and the $\hat{x}$-axis. In addition, the function f is the product of two elements of the spectral domain DGFs and thus usually contains the arguments of $(k_x/k_\rho)^n$ or $(k_y/k_\rho)^n$ which can be easily transformed into the expressions of $\cos(n\xi)$ or $\sin(n\xi)$. As a result, (A2) can be further simplified using the following identity:

$$\frac{1}{2\pi}\int_0^{2\pi} \begin{matrix}\cos\\ \sin\end{matrix} n\xi e^{-j\boldsymbol{k_\rho}\cdot\boldsymbol{\rho}\cos(\xi-\phi)} d\xi = (-j)^n J_n(k_\rho\rho) \begin{matrix}\cos\\ \sin\end{matrix} n\phi \quad (A3)$$

where J_n is the nth-order Bessel function.

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